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SEMANTICAL INVESTIGATIONS IN HEYTING'S INTUITIONISTIC LOGIC

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CHAPTER 1

LOGICAL SYSTEMS AND SEMANTICS

This chapter discusses the notions of a logical system, a semantics for a logical system, and the notion of what is a classical connective in a logical system. Examples are given, to prepare the background for the introduction of the Heyting systems in the next chapter.

1. Scott and Tarski Systems

DEFINITION 1. Let L be a language, and let φ, ψ denote finite, possibly empty sets of wffs. Let $\emptyset$ denote the empty set. A binary relation $\Vdash$ on sets φ, ψ is called *Scott consequence relation* iff the following conditions hold:

(a) $\varphi \Vdash \varphi$, for $\varphi \neq \emptyset$

(b) if $\varphi \Vdash \psi$ then $\varphi \cup \varphi' \Vdash \psi \cup \psi'$ for any φ', ψ'.

(c) (Cut rule): if $\varphi, A \Vdash \psi$ and $\varphi \Vdash \psi, A$ then $\varphi \Vdash \psi$

DEFINITION 2. We write $\varphi, A \Vdash \psi, B$ instead of $\varphi \cup \{A\} \Vdash \psi \cup \{B\}$. Similarly, we use $\varphi, \varphi' \Vdash \psi, \psi'$ and $\varphi, A_1, \ldots, A_n \Vdash \psi, B_1, \ldots, B_k$, instead of $\varphi \cup \varphi' \Vdash \psi \cup \psi'$ and $\varphi \cup \{A_1, \ldots, A_n\} \Vdash \psi \cup \{B_1, \ldots, B_k\}$ respectively.

DEFINITION 3. (a) A (Scott) consequence relation is said to be *consistent* iff $\emptyset \nVdash \emptyset$.

(b) A Scott consequence relation $\Vdash$ is said to be a *Scott system* (S.S) iff $\Vdash$ is closed under substitution.

DEFINITION 4. Let Δ, Θ be sets of wffs, and let $\Vdash$ be a Scott consequence relation. We write $\Delta \Vdash \Theta$ iff for some $\varphi \subseteq \Delta$, $\psi \subseteq \Theta$, $\varphi \Vdash \psi$.

Exercise 5. Let Δ, Θ be sets of wffs such that $\Delta \nVdash \Theta$. Define a relation $\Vdash^*$ by $\varphi \Vdash^* \psi$ iff $\Delta, \varphi \Vdash \Theta, \psi$. Show $\Vdash^*$ is a Scott consequence relation.

LEMMA 6. *For any Scott consequence relation $\Vdash$, if $\varphi, A_i \Vdash \psi$ for $1 \leq i \leq n$ and $\varphi \Vdash \psi, A_1, \ldots, A_n$ then $\varphi \Vdash \psi$.*

Proof. By induction on n. For $n = 1$ this is the cut rule. For $n = k + 1$, notice that $\varphi, A_i \Vdash \psi, A_{k+1}$ for $1 \leqslant i \leqslant k$ and $\varphi \Vdash \psi, A_{k+1}, A_1, \dots, A_k$ and so by the induction hypothesis $\varphi \Vdash \psi, A_{k+1}$. Now since $\varphi, A_{k+1} \Vdash \psi$ we conclude $\varphi \Vdash \psi$.

DEFINITION 7. A *Tarski consequence relation* (for L) is a binary relation containing pairs of the form (φ, ψ) (written $\varphi \vdash \psi$) with $\bar{\bar{\psi}} = 1$, satisfying the following properties. (We use the conventions of Definition 2 for $\vdash$ as well.)

(a) $A \vdash A$

(b) if $\varphi \vdash \psi$ then $\varphi, \varphi' \vdash \psi$

(c) if $\varphi, C \vdash \psi$ and $\varphi \vdash C$ then $\varphi \vdash \psi$ (cut rule)

DEFINITION 8. (a) A Tarski consequence relation $\vdash$ is called a *Tarski system* (T.S) iff $\vdash$ is closed under substitution.

(b) $\vdash$ is said to be consistent iff for some φ, A, $\varphi \nvdash A$.

Exercise 9. Let $\Vdash$ be a Scott consequence relation. For $\bar{\bar{\psi}} = 1$ let $\varphi \vdash \psi$ iff (def) $\varphi \Vdash \psi$; show that $\vdash$ is a Tarski consequence relation.

LEMMA 10. *If* $\varphi \vdash A_i$, $1 \leqslant i \leqslant n$ *and* $\varphi, A_1, \dots, A_n \vdash \psi$ *then* $\varphi \vdash \psi$.

Proof. Let $\varphi' \subseteq \{A_1, \dots, A_n\}$ and let $\bar{\bar{\varphi}}' = n - k$. We show by induction on k that $\varphi \cup \varphi' \vdash \psi$. The lemma will follow for the case $n = k$.

Case $k = 1$: Let $\{A_1, \dots, A_n\} = \varphi' \cup \{A\}$. $A \notin \varphi'$. Then $\varphi \cup \varphi' \vdash A$ and $\varphi, \varphi', A \vdash \psi$ and therefore by the cut rule, $\varphi, \varphi' \vdash \psi$.

Case k: Let $\varphi'' = \varphi' \cup \{A\}$, $\bar{\bar{\varphi}}'' = n - k$, with $A \notin \varphi'$, $\varphi'' \subseteq \{A_1, \dots, A_n\}$.

By the induction hypothesis $\varphi, \varphi' \vdash \psi$ but also $\varphi, \varphi' \vdash A$ and therefore by the cut rule, $\varphi, \varphi' \vdash \psi$. This proves Lemma 10.

THEOREM 11. *Let* $\vdash$ *be a Tarski consequence relation and let* $\Vdash^-$ *be defined by* $\varphi \Vdash^- \psi$ *iff (def) for some* $B \in \psi$, $\varphi \vdash B$, *then* $\Vdash^-$ *is a Scott consequence relation.*

Proof. Clearly conditions (a) and (b) of Definition 1 are satisfied. We verify the cut rule. Assume that $\varphi, C \Vdash^- \psi$ and $\varphi \Vdash^- \psi, C$. By definition, for some $B \in \psi$ and $A \in \psi \cup \{C\}$ we have that $\varphi, C \vdash B$

and $\varphi \vdash A$. If $A \in \psi$ we are finished. If $A \notin \psi$, then $A = C$, and thus $\varphi \vdash C$, and $\varphi, C \vdash B$ and so $\varphi \vdash B$ and again we are finished.

DEFINITION 12. Let $\vdash$ be a Tarski consequence relation and let $\Vdash$ be a Scott consequence relation (for the same language L). $\Vdash$ and $\vdash$ are said *to agree* iff for all φ, ψ, $\bar{\psi} = 1$, $\varphi \vdash \psi$ iff $\varphi \Vdash \psi$.

THEOREM 13 (*Scott*). *Let $\vdash$ be a Tarski consequence relation, then there exist two Scott consequence relations $\Vdash^{+}_{\vdash}$ and $\Vdash^{-}_{\vdash}$ that agree with $\vdash$ and such that for any $\Vdash$ that agrees with $\vdash$ we have $\Vdash^{-}_{\vdash} \subseteq \Vdash \subseteq \Vdash^{+}_{\vdash}$.*

Proof. (a) For $\Vdash^{-}_{\vdash}$ take the Scott consequence relation defined in Theorem 11. Assume that $\Vdash$ is any Scott consequence relation that agrees with $\vdash$. Then if $\varphi \Vdash^{-}_{\vdash} \psi$ then for some $A \in \psi$, $\varphi \vdash A$ and therefore $\varphi \Vdash A$ and hence $\varphi \Vdash \psi$.

(b) We define $\Vdash^{+}_{\vdash}$.

Let $\varphi \Vdash^{+}_{\vdash}$ iff for some finite set Δ, property $(*)$ below holds, Where:

$(*)$ For any partition (Δ_1, Δ_2) of Δ (i.e. $\Delta_1 \cup \Delta_2 = \Delta$, $\Delta_1 \cap \Delta_2 = \emptyset$) there exists a Scott consequence relation $\Vdash$ that agrees with $\vdash$ such that $\varphi, \Delta_1 \Vdash \psi, \Delta_2$.

First we show that $\Vdash^{+}_{\vdash}$ is a consequence relation

(a) Clearly if $\varphi \neq \emptyset$ then $\varphi \Vdash^{+}_{\vdash} \varphi$ take $\Delta = \emptyset$

(b) If $\varphi \Vdash^{+}_{\vdash} \psi$ let Δ be such that $(*)$ holds, then for any φ', ψ', $\varphi \cup \varphi' \Vdash^{+}_{\vdash} \psi \cup \psi'$ as the same Δ is adequate.

(c) Assume $\varphi, A \Vdash^{+}_{\vdash} \psi$ and $\varphi \Vdash^{+}_{\vdash} \psi, A$. Let Δ, Δ^* resp. be the two sets having the properties $(*)$ in the definition of $\Vdash^{+}_{\vdash}$. Regard $\Delta' = \Delta \cup \Delta^* \cup \{A\}$, we claim $\varphi \Vdash^{+}_{\vdash} \psi$ since Δ' has the property $(*)$ required in the definition.

We now have to show that $\Vdash^{+}_{\vdash}$ agrees with $\vdash$. Assume $\varphi \Vdash^{+}_{\vdash} A$ we want to show that $\varphi \vdash A$. Since $\varphi \Vdash^{+}_{\vdash} A$, there exists a Δ with property $(*)$. We show by induction on n, that for any $\Theta \subseteq \Delta$, $\bar{\bar{\Theta}} = n$ we have that $\varphi \cup (\Delta - \Theta) \vdash A$. For $n = 1$, let $B \in \Theta$, be arbitrary. So $\varphi \cup (\Delta - \{B\}) \Vdash_1 A, B$ for some $\Vdash_1$, that agrees with $\vdash$, because property $(*)$ holds. Also for some $\Vdash_2$ that agrees with $\vdash$, $\varphi, \Delta \Vdash_2 A$. Since $\Vdash_1$, $\Vdash_2$ agree with $\vdash$ we get that $\varphi, \Delta \Vdash_1 A$ and so by cut, $\varphi \cup (\Delta - \{B\}) \Vdash_1 A$ and so $\varphi \cup (\Delta - \{B\}) \vdash A$.

Now assume that for any Θ, $\bar{\bar{\Theta}} \leq m$, $\varphi \cup (\Delta - \Theta) \vdash A$, show this for any Θ, $\bar{\bar{\Theta}} = m + 1$. Let such Θ be given then for any $B \in \Theta$, $\varphi \cup$

$(\Delta - \Theta), B \vdash A$. Also by property $(*)$, there exists a $\Vdash$ that agrees with $\vdash$ such that $\varphi, \Delta - \Theta \Vdash \Theta, A$. Now since $\Vdash$ agrees with $\vdash$ we get by Lemma 6 that $\varphi, \Delta - \Theta \Vdash A$, and therefore $\varphi, \Delta - \Theta \vdash A$. This completes the induction step. If we take $\Theta = \Delta$, we get that $\varphi \vdash A$, and thus we see that $\Vdash^+_\vdash$ agrees with $\vdash$.

To show that if $\Vdash$ agrees with $\vdash$ then $\Vdash \subseteq \Vdash^+_\vdash$, assume that $\varphi \Vdash \psi$, then for Δ empty we get property $(*)$ and so $\varphi \Vdash^+_\vdash \psi$.

Exercise 14. Let $\vdash$ be a Tarski consequence relation and let $\Vdash^+_\vdash$ be the maximal Scott consequence relation agreeing with $\vdash$. Let $con(\Delta)$ be $con(\Delta) = \{A \mid \text{for some } \varphi \subseteq \Delta, \varphi \vdash A\}$. Show that:

$$\varphi \Vdash^+_\vdash \psi \quad \text{iff} \quad \bigcap_{B \in \psi} con(\varphi' \cup \{B\}) \subseteq con(\varphi')$$

for all $\varphi' \supseteq \varphi$.

DEFINITION 15. (a) A Hilbert (or axiomatic) system H is a triple (H_0, H_1, H_2) where H_0 is a set of wffs called axioms, H_1 is a set of rules of the form $A_1, \ldots, A_n/B$, called provability rules and H_2 is a set of rules of the form φ/ψ, with $\bar{\bar{\psi}} = 1$, called consequence rules.

(b) Given a Hilbert system H, we define the relation $\vdash_H A$, on wff A as follows: $\vdash_H A$ iff there exists a finite sequence of wff $B_1, \ldots, B_k = A$ such that each B_i of the sequence is either a substitution instance of a member of H_0 or for some wffs $A_1, \ldots, A_n$, appearing earlier than B_i in the sequence, we have that $A_1, \ldots, A_n/B_i$ is a rule of H_1.

(c) Given a Hilbert system H we define the notion $\varphi \vdash_H \psi$, for $\bar{\bar{\psi}} = 1$ as follows: $\varphi \vdash_H \psi$ iff there exists a sequence of wff $B_1, \ldots, B_n$ such that (i) and (ii) below hold:

(i) For each $i \leqslant n$ either (1) $B_i \in \varphi$ or (2) $\vdash_H B_i$ ($\vdash_H$ of (b) above) or (3). For some $A_1, \ldots, A_k$ appearing earlier in the sequence $\{A_1, \ldots, A_k\}/\{B_i\}$ is a substitution instance of a rule of H_2.

(ii) Either (1) $\psi = \{B\}$ with $B \in \varphi$ or (2) $\{B_1, \ldots, B_n\}/\psi$ is a substitution instance of a rule of H_2.

Remark. We use the abbreviations of Definition 2 for Hilbert systems as well.

THEOREM 16. *Let H be a Hilbert system, then $\vdash_H$ is a Tarski system.*

Proof. Let us check the cut rule. Assume φ, $C \vdash_H \psi$ and $\varphi \vdash_H C$ we must show that $\varphi \vdash_H \psi$. Let $B_1, \dots, B_n$ be a proof of C from φ, and let $A_1, \dots, A_k$ be a proof of ψ from $\varphi \cup \{C\}$. Then the following is a proof of ψ from φ: $B_1, \dots, B_n, C, A_1, \dots, A_k$. It is easy to verify that $\vdash_H$ is closed under substitution.

THEOREM 17. *Let $\vdash$ be a Tarski system, then there exists a Hilbert system H such that $\vdash = \vdash_H$.*

Proof. Let H be the Hilbert system (H_0, H_1, H_2) with $H_0 = \{B \mid \emptyset \vdash B\}$, $H_1 = \emptyset$, $H_2 = \{\varphi/\psi \mid \varphi \vdash \psi\}$; clearly $\vdash \subseteq \vdash_H$. We want to show that $\vdash_H \subseteq \vdash$. Assume $\varphi \vdash_H \psi$. Let $B_1, \dots, B_n$ be a proof of ψ from φ. We show by induction on i that $\varphi \vdash B_i$. If $B_i \in \varphi$ this is clear. If B_i is obtained from some $A_1, \dots, A_k$ appearing previously in the sequence then by the induction hypothesis $\varphi \vdash A_j$ and also by the definition of H_2, $A_1, \dots, A_k \vdash B_i$ therefore by Lemma 10, $\varphi \vdash B_i$. Now ψ is obtained by clause (15c3ii) i.e. either $\psi = \{B\} \subseteq \varphi$, in which case $\varphi \vdash B$ or $\{B_1, \dots, B_n\}/\psi$ is a rule of H_2, i.e. $B_1, \dots, B_n \vdash \psi$, so again $\varphi \vdash \psi$ by Lemma 10.

DEFINITION 18. Let $\vdash$ be a Tarski consequence relation and let $\Vdash_{\vdash}^{-}$ and $\Vdash_{\vdash}^{+}$ be the minimal and maximal Scott consequence relations that agree with $\vdash$. Define $S_{\vdash}$, called *the slash* of $\vdash$ by:

$$S_{\vdash} = \{\varphi \mid \text{for all } \psi, \quad \varphi \Vdash_{\vdash}^{+} \psi \text{ implies } \varphi \Vdash_{\vdash}^{-} \psi.\}$$

Remark. If $\emptyset \in S_{\vdash}$, then $\vdash$ has the 'disjunction property' in a certain sense. We shall return to this notion later.

2. Scott Semantics 1

DEFINITION 1. Let $\Vdash$ be a Scott consequence relation.

(a) By *a theory* we mean a pair (Δ, Θ) of sets of wffs.

(b) (Δ, Θ) is said to be $\Vdash$-*consistent* iff $\Delta \not\Vdash \Theta$.

(c) (Δ, Θ) is said to be *complete* iff $\Delta \cup \Theta$ is the set of all wffs.

(d) (Δ', Θ') is said to *extend* (Δ, Θ) iff $\Delta \subseteq \Delta'$, $\Theta \subseteq \Theta'$.

DEFINITION 2. (a) By a *model* we mean a function $\mathbf{t}$ assigning a value in $\{0, 1\}$ to each wff of L.

(b) A *semantics* $\mathbf{T}$ is a set of models.

DEFINITION 3. Let $\mathbf{T}$ be a semantics.

Define: $\varphi \Vdash_{\mathbf{T}} \psi$ iff for all $\mathbf{t} \in \mathbf{T}$ the following holds: If for all $A \in \varphi$, $\mathbf{t}(A) = 1$, then for some $B \in \psi$, $\mathbf{t}(B) = 1$.

LEMMA 4. *$\Vdash_{\mathbf{T}}$ is a Scott consequence relation.*

Proof. Exercise.

LEMMA 5. *Let (Δ, Θ) be $\Vdash$ consistent. Then there exists a $\Vdash$ consistent and complete extension (Δ', Θ') of (Δ, Θ).*

Proof. Let $A_1, A_2, A_3, \ldots$ be an enumeration of all the wffs of L. Define by induction a sequence (Δ_n, Θ_n) of $\Vdash$-consistent theories such that for all n, $\Delta \subseteq \Delta_n \subseteq \Delta_{n+1}$, $\Theta \subseteq \Theta_n \subseteq \Theta_{n+1}$. Let $\Delta_0 = \Delta$, $\Theta_0 = \Theta$. Assume (Δ_n, Θ_n) has been defined. Regard A_n, if Δ_n, $A_n \nVdash \Theta_n$, let $\Delta_{n+1} = \Delta_n \cup \{A_n\}$, $\Theta_{n+1} = \Theta_n$. If Δ_n, $A_n \Vdash \Theta_n$, then $\Delta_n \nVdash \Theta_n, A_n$, since otherwise we can get $\Delta_n \Vdash \Theta_n$, contrary to the inductive hypotheses. So let $\Delta_{n+1} = \Delta_n$, $\Theta_{n+1} = \Theta_n \cup \{A_n\}$. Thus $(\Delta_{n+1}, \Theta_{n+1})$ is defined and is $\Vdash$ consistent in either case.

Now let $\Delta' = \bigcup_n \Delta_n$, $\Theta' = \bigcup_n \Theta_n$.

It is easy to show that (Δ', Θ') is the desired extension.

DEFINITION 6. (a) Let (Δ, Θ) be a $\Vdash$ consistent and complete theory. Let $\mathbf{t}_{(\Delta,\Theta)}$ be the model with $\mathbf{t}_{(\Delta,\Theta)}(A) = 1$ iff $A \in \Delta$.

(b) Let $\mathbf{T}^*_{\Vdash}$ be the semantics with

$$\mathbf{T}^*_{\Vdash} = \{\mathbf{t}_{(\Delta,\Theta)} \mid (\Delta, \Theta) \Vdash \text{complete and consistent}\}.$$

THEOREM 7. (*Scott completeness theorem*). $\Vdash\ = \ \Vdash_{\mathbf{T}^*_{\Vdash}}$.

Proof. $\varphi \Vdash \psi$ iff (by Lemma 5) no $\Vdash$ complete and consistent theory (Δ, Θ) extends (φ, ψ) iff (by definition) $\varphi \Vdash_{\mathbf{T}^*_{\Vdash}} \psi$.

Exercise 8. Let $\Vdash$ be Scott consequence relation, and let (Δ, Θ) be a $\Vdash$-consistent theory. Define $\Vdash^*$ by $\varphi \Vdash^* \psi$ iff $\Delta, \varphi \Vdash \Theta, \psi$. Show that $\Vdash^*$ is a Scott consequence relation.

3. What is a Classical Connective?

DEFINITION 1. Let $\Vdash$ be a consequence relation and let $f \in 2^{2^n}$. Let $\#$ be an n-place connective in the language of $\Vdash$.

Consider the following set of conditions (denoted by R_f) on $\Vdash$.

For each $\bar{a} \in \{0, 1\}^n$ take $\#\bar{a}$:

$$\#\bar{a}\colon\ \varphi_{\bar{a}} \Vdash \psi_{\bar{a}}$$

Where $\varphi_{\bar{a}}, \psi_{\bar{a}} \subseteq \{A_0, \dots, A_{n-1}, \#(A_0, \dots, A_{n-1})\}$ have the property that

$$\begin{aligned}
\bar{a}(i) = 1 &\quad \text{iff} \quad A_i \in \varphi_{\bar{a}}\\
\bar{a}(i) = 0 &\quad \text{iff} \quad A_i \in \psi_{\bar{a}}\\
f(\bar{a}) = 1 &\quad \text{iff} \quad \#(A_0, \dots, A_{n-1}) \in \psi_{\bar{a}}\\
f(\bar{a}) = 0 &\quad \text{iff} \quad \#(A_0, \dots, A_{n-1}) \in \varphi_{\bar{a}}
\end{aligned}$$

DEFINITION 2. Let $\Vdash$ be a Scott consequence relation for a language with the n-ary connective $\#$. We say the $\#$ is classical in $\Vdash$ with truth table f iff all the conditions of R_f hold for $\Vdash$, for any A_i.

When we turn to Tarski consequence relations $\vdash$, the problem of which connectives are classical is more difficult. One may give the following definition.

DEFINITION 3. Let $\vdash$ be a Tarski consequence relation for a language with the connective $\#$. We say that $\#$ is strongly classical in $\vdash$ with truth table f iff for every Scott consequence relation $\Vdash$ agreeing with $\vdash$ we have that $\#$ is classical in $\Vdash$ with truth table f.

DEFINITION 4. Let $\vdash$ be a Tarski consequence relation for a language L and $\#$ be a connective of L. Then $\#$ is said to be weakly classical with truth table f iff there exists a Scott consequence relation $\Vdash$ agreeing with $\vdash$ in which $\#$ is classical with truth table f.

THEOREM 5. *Let* $\vdash$ *be a consistent Tarski system in a language with the n-ary connective* $\#$. *Then* $\#$ *is strongly classical in* $\vdash$ *iff either* (a) $\emptyset \vdash \#$ *or* (b) *for some* $B_1, \dots, B_k \in \{A_1, \dots, A_n\}$, $B_1, \dots, B_k \vdash \#(A_1, \dots, A_n)$ *and* $\#(A_1, \dots, A_n) \vdash B_i$, *for each* $1 \leqslant i \leqslant k$, *where* $A_1, \dots, A_n$ *are arbitrary atomic wffs.*

Remark. Theorem 5 says that essentially only conjunctions can be strongly classical in $\vdash$.

Proof. Since $\#$ is strongly classical in $\vdash$, it is classical in $\Vdash^{-}_{\vdash}$, the minimal consequence relation agreeing with $\vdash$. Let f be a truth table, with regard to which $\#$ is classical in $\Vdash^{-}_{\vdash}$. Let $T \subseteq \{1, \dots, n\}$. Let $\bar{a}_T$ be such that $\bar{a}_T(j) = 1$ iff $j \in T$. We proceed to show that the theorem holds. Let $A_1, \dots, A_n$ be atomic, and ask what is the value $f(\bar{a}_\emptyset)$? If the value is 1 then $\emptyset \Vdash^{-}_{\vdash} A_1, \dots, A_n, \#(A_1, \dots, A_n)$, and since A_i are atomic and $\vdash$ consistent, we must have $\emptyset \vdash \#(A_1, \dots, A_n)$, which is case (a) of the theorem.

Otherwise, $f(\bar{a}_\emptyset) = 0$, and so $\#(A_1, \dots, A_n) \vdash A_1, \dots, A_n$ and therefore for some $j \in \{1, \dots, n\}$, $\#(A_1, \dots, A_n) \vdash A_j$. Regard $\bar{a}_{\{j\}}$, if $f(\bar{a}_{\{j\}}) = 1$, then $A_j \Vdash^{-}_{\vdash} \{A_i \mid i \neq j\}, \#(A_1, \dots, A_n)$ and again since A_i are atomic, $A_j \vdash \#(A_1, \dots, A_n)$ and thus case (b) of the theorem holds.

Otherwise, $f(\bar{a}_{\{j\}}) = 0$ and so $A_j, \#(A_1, \dots, A_2) \Vdash^{-}_{\vdash} \{A_i \mid i \neq j\}$. Since $\#(A_1, \dots, A_n) \vdash A_j$, we get that $\#(A_1, \dots, A_n) \Vdash^{-}_{\vdash} \{A_i \mid i \neq j\}$.

Assume by induction on $1 \leq i \leq n$, that there exists a $T \subseteq \{1, \dots, n\}$, $\bar{\bar{T}} = i$ with the property that either (1) For some $B_{j_1}, \dots, B_{j_k}$ $j_1, \dots, j_k \in T$ case (b) of the theorem holds or (2) $\#(A_1, \dots, A_n) \vdash B_j$ for each $j \in T$.

We find such a T' with $\bar{\bar{T}}' = i + 1$. If case (1) holds, any element can be added to T to form T'. If case (2) holds, consider $\bar{a}_T$. If $f(\bar{a}_T) = 1$, then $\{A_j \mid j \in T\} \Vdash^{-}_{\vdash} \{A_j \mid j \notin T\}, \#(A_1, \dots, A_n)$ and since A_i are atomic $\{A_j \mid j \in T\} \vdash \#(A_1, \dots, A_n)$ which yields case (1) for T. If $f(\bar{a}_T) = 0$, we get $\{A_j \mid j \in T\}, \#(A_1, \dots, A_n) \Vdash^{-}_{\vdash} \{A_j \mid j \notin T\}$, let $j_0 \notin T$ be such that $\{A_j \mid j \in T\}, \#(A_1, \dots, A_n) \vdash A_{j_0}$. Since $\#(A_1, \dots, A_n) \vdash A_j$ for $j \in T$ we get by Lemma 1.10 that $\#(A_1, \dots, A_n) \vdash A_{j_0}$. This yields case (2) for $T' = T \cup \{j_0\}$.

Now consider the case of $i = n$. If the case (1) holds, then case (b) of the theorem is valid. If case (2) holds, then $\# \vdash A_i$, $1 \leq i \leq n$. Consider $\bar{a}_T$ for $T = \{1, \dots, n\}$. If $f(a_T) = 1$ we get that $A_1, \dots, A_n \vdash \#(A_1, \dots, A_n)$ which yields case (b) of the theorem. If $f(a_T) = 0$, we get $A_1, \dots, A_n, \#(A_1, \dots, A_n) \Vdash^{-}_{\vdash} \emptyset$ and since $\# \vdash A_i$, $1 \leq i \leq n$, we get $\# \Vdash^{-}_{\vdash} \emptyset$ which is impossible by the definition of $\Vdash^{-}_{\vdash}$. Thus theorem 5 is proved.

Exercise 6. Let $\Vdash$ be a Scott consequence relation and let $\#(A_1, \dots, A_n)$ be an n-ary connective. Show that $\#$ is classical in $\Vdash$ with truth table f iff for all $\mathbf{t} \in T_{\Vdash}$, $\mathbf{t}(\#(A_1, \dots, A_n) = f(\mathbf{t}(A_1), \dots, \mathbf{t}(A_n))$.

Exercise 7. Let $\vdash$ be a Tarski consequence relation and $\Vdash^-_{\vdash}$, $\Vdash^+_{\vdash}$ be the maximal and minimal Scott consequence relations that agree with it. Show that:

(a) If $\vdash$ is a Tarski system then $\Vdash^-_{\vdash}$, $\Vdash^+_{\vdash}$ are Scott systems.

(b) If the connective $\#$ is weakly classical in $\vdash$ (with truth table f) then it is classical in $\Vdash^+_{\vdash}$ (with truth table f).

(c) If the connective $\#$ is strongly classical in $\vdash$ (with truth table f) then it is classical in $\Vdash^-_{\vdash}$ (with truth table f).

COROLLARY. *If $\#_i$ are weakly classical in $\vdash$, for $1 \leqslant i \leqslant n$, then for some $\Vdash$ that agrees with $\vdash$, $\#_i$, $1 \leqslant i \leqslant n$ are all classical in $\Vdash$.*

Exercise 8. Let $\Vdash$ be a Scott consequence relation in a language with some or all of the following connectives:

t, f	zero place
$\sim$	one place
$\wedge, \vee, \rightarrow$	two place

Show that these connectives have their respective classical table in $\Vdash$ iff the following holds (respectively), for all A, B.

(1) $\emptyset \Vdash t$

(2) $f \Vdash \emptyset$

(3) $A \wedge B \Vdash A;\quad A \wedge B \Vdash B;\quad A, B \Vdash A \wedge B$

(4) $A \Vdash A \vee B;\quad B \Vdash A \vee B;\quad A \vee B \Vdash A, B$

(5) $A, \sim A \Vdash \emptyset;\quad \emptyset \Vdash A, \sim A$

(6) $A, A \rightarrow B \Vdash B;\quad \emptyset \Vdash A, A \rightarrow B.$

DEFINITION 9. Let $\Vdash$ be a Scott consequence relation and let Q be a unary quantifier of the language. We say that Q is the classical universal quantifier in $\Vdash$ iff the following always holds for all A, φ, ψ.

(a) $(Qx)A(x) \Vdash A(y)$

(b) $\varphi \Vdash A(x), \psi$ iff $\varphi \Vdash (Qx)A(x), \psi$

where x does not appear free in any wff $\varphi \cup \psi$.